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A.A. Pavlov

OPTIMIZATION FOR ONE CLASS OF COMBINATORIAL PROBLEMS UNDER UNCERTAINTY

Abstract: We formalize uncertainty concept, compromise criteria of uncertainty minimization, and efficient formal procedures for a sufficiently common class of combinatorial optimization problems which functional contains numerical parameters. The procedures are based on the well-known idea of linear convolution of criteria. They optimize or implement the compromise criteria or conditions we propose.

Keywords: Combinatorial optimization, uncertainty, probability, efficient algorithm, uncertainty resolution, PSC-algorithm, NP-hard problems.

Introduction

Definitions:

Let σ be a feasible solution;

Ω be a set of feasible solutions of an arbitrary form. One can specify Ω by arbitrary constraints, conditions, enumeration of feasible solutions, etc.

We study the class of combinatorial optimization problems of the following form:

$$\min_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_i k_i(\sigma) \tag{1}$$

where ω_i are numbers, $k_i(\sigma)$ is i-th arbitrary numerical characteristic of a feasible solution σ.

The model (1) is sufficiently common. It includes, in particular, the following combinatorial optimization problems: transportation problem, flow network problems, NP-hard combinatorial optimization problems (e.g., the total weighted tardiness of tasks minimization on one machine [1], the total weighted completion time of interrelated tasks minimization on one machine [1]) etc.

Remark 1. Efficient PSC-algorithms exist [2] to solve these NP-hard combinatorial problems.

We will consider two formulations of the problem (1) under uncertainty. The first formulation of the problem is partial, the second one is more general. This differentiation is explained by the fact that we were able to obtain some results within the framework of the first formulation of the combinatorial optimization problem under uncertainty that were absent within the framework of the more general second formulation of the problem.

The first formulation of the combinatorial optimization problem under uncertainty

Under the uncertainty of the problem (1) we will understand the uncertainty of the values of coefficients ω_i, $i=\overline{1,s}$, at the time of the problem (1) solving.

$\overline{\omega} = (\omega_1, \ldots, \omega_s)$ can take one of two possible values: $\overline{\omega_1} = (\omega_{11}, \ldots, \omega_{1s})$ or $\overline{\omega_2} = (\omega_{21}, \ldots, \omega_{2s})$. Thus, the stage of the problem (1) solving and the stage of its solution implementation may be separated in time. For example, there may be the forward planning stage and the plan execution stage when changes of an external environment may affect the values of the coefficients ω_i, $i = \overline{1, s}$ determined at the stage of the forward planning.

We can specify probabilities $P_1 > 0$, $P_2 = 1 - P_1 > 0$ (the probabilities are absent if the uncertainty is not described by a probabilistic model).

To solve the problem (1) under uncertainty means to find a feasible solution that satisfies one of the following conditions:

1) σ corresponds to $min(\Delta_1 + \Delta_2)$ where

$$\Delta_1 = \sum_{i=1}^{s} \omega_{1i} k_i(\sigma) - f_{opt}^1, \quad \Delta_2 = \sum_{i=1}^{s} \omega_{2i} k_i(\sigma) - f_{opt}^2,$$

$$f_{opt}^1 = \min_{\sigma \in \Omega} \sum_{i=1}^{s} \omega_{1i} k_i(\sigma), \quad f_{opt}^2 = \min_{\sigma \in \Omega} \sum_{i=1}^{s} \omega_{2i} k_i(\sigma).$$

2) σ satisfies the condition $\Delta_1 \le l_1$, $\Delta_2 \le l_2$ where $l_1 > 0$, $l_2 > 0$ and are specified by experts.

3) $min\, \Delta_1$, $\Delta_2 \le l_2$ or $min\, \Delta_2$, $\Delta_1 \le l_1$.

4) $min(a_1\Delta_1 + a_2\Delta_2)$ where $a_1 > 0$, $a_2 > 0$ are specified by experts.

5) $\min_{\sigma \in \Omega} \{M\Delta = P_1\Delta_1 + P_2\Delta_2\}$ where M is mathematical expectation operator, if the uncertainty is specified with a probabilistic model.

The criterion to find a compromise solution is specified by experts.

Finding a compromise solution for each of the five criteria (conditions)

We first investigate the theoretical properties of the first model of the combinatorial optimization problem (1) under uncertainty.

Statement 1. It is true that

$$\arg \min_{\sigma \in \Omega} \left\{ a_1 \left[\sum_{i=1}^{s} \omega_{1i} k_i(\sigma) - f_{opt}^1 \right] + a_2 \left[\sum_{i=1}^{s} \omega_{2i} k_i(\sigma) - f_{opt}^2 \right] \right\} =$$

$$= \arg \min_{\sigma \in \Omega} \sum_{i=1}^{s} (a_1\omega_{1i} + a_2\omega_{2i}) k_i(\sigma), \quad a_1 > 0, a_2 > 0.$$

Proof. Let us denote

$$\{\sigma_1\} = \arg \min_{\sigma \in \Omega} \sum_{i=1}^{s} \omega_{1i} k_i(\sigma); \tag{2}$$

$$\{\sigma_2\} = \arg \min_{\sigma \in \Omega} \sum_{i=1}^{s} \omega_{2i} k_i(\sigma); \tag{3}$$

σ_1 is the element of the set $\{\sigma_1\}$ at which we reach $\min_{\sigma \in \{\sigma_1\}} \left[\sum_{i=1}^{s} \omega_{2i} k_i(\sigma) - f_{opt}^1 \right]$. $\sigma_2 \in \{\sigma_2\}$, and we reach $\min_{\sigma \in \{\sigma_2\}} \left[\sum_{i=1}^{s} \omega_{1i} k_i(\sigma) - f_{opt}^2 \right]$ at σ_2 (we consider algorithms

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of σ_1, σ_2 obtaining in the second framework of uncertainty for the case when Ω is finite);

$$\sigma^1(a_1, a_2) = arg\min_{\sigma\in\Omega}\left\{a_1\left[\sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1\right] + a_2\left[\sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2\right]\right\}; \quad (4)$$

$$\sigma^2(a_1, a_2) = arg\min_{\sigma\in\Omega}\sum_{i=1}^{s}(a_1\omega_{1i} + a_2\omega_{2i})k_i(\sigma), \quad a_1 > 0, a_2 > 0 \quad (5)$$

where $\sigma^1(a_1, a_2)$ is the set of solutions of (4), and $\sigma^2(a_1, a_2)$ is the set of solutions of (5). We have:

$$\sum_{i=1}^{s}\omega_{1i}k_i(\sigma^1(a_1, a_2)) - f_{opt}^1 \ge 0, \quad \sum_{i=1}^{s}\omega_{2i}k_i(\sigma^1(a_1, a_2)) - f_{opt}^2 \ge 0; \quad (6)$$

$$a_1\sum_{i=1}^{s}\omega_{1i}k_i(\sigma^1(a_1, a_2)) - a_1 f_{opt}^1 \ge 0, \quad a_2\sum_{i=1}^{s}\omega_{2i}k_i(\sigma^1(a_1, a_2)) - a_2 f_{opt}^2 \ge 0; \quad (7)$$

$$a_1\left[\sum_{i=1}^{s}\omega_{1i}k_i(\sigma^1(a_1, a_2)) - f_{opt}^1\right] + a_2\left[\sum_{i=1}^{s}\omega_{2i}k_i(\sigma^1(a_1, a_2)) - f_{opt}^2\right] =$$

$$= \sum_{i=1}^{s}(a_1\omega_{1i} + a_2\omega_{2i})k_i(\sigma^1(a_1, a_2)) - (a_1 f_{opt}^1 + a_2 f_{opt}^2) \text{ and} \quad (8)$$

$$\sum_{i=1}^{s}(a_1\omega_{1i} + a_2\omega_{2i})k_i(\sigma^1(a_1, a_2)) - (a_1 f_{opt}^1 + a_2 f_{opt}^2) \ge 0. \quad (9)$$

Remark 2. (6)–(9) are true for $\forall\sigma \in \sigma^1(a_1, a_2)$.

Inequalities (6), (7), (9) also hold for $\forall\sigma\in\Omega$ and therefore for $\forall\sigma \in \sigma^2(a_1, a_2)$. Since $a_1 f_{opt}^1 + a_2 f_{opt}^2 = const$, we reach the minimum $\min_{\sigma\in\Omega}\sum_{i=1}^{s}(a_1\omega_{1i} + a_2\omega_{2i})k_i(\sigma)$ for $\forall\sigma \in \sigma^1(a_1, a_2)$ and, consequently, we reach $\min_{\sigma\in\Omega}\left\{a_1\left[\sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1\right] + a_2\left[\sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2\right]\right\}$ for $\forall\sigma \in \sigma^2(a_1, a_2)$. Proven.

Corollary 1. We have reduced the problem

$$\min_{\sigma\in\Omega}\left\{a_1\left[\sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1\right] + a_2\left[\sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2\right]\right\} \quad (10)$$

solving to the problem (1) solving with the functional coefficients $\omega_i = a_1\omega_{1i} + a_2\omega_{2i}$, $i = \overline{1, s}$. Thus, if there is an efficient algorithm for the problem (1) solving, then this algorithm automatically solves the problem (10) efficiently.

Corollary 2. Suppose $L_1 = \sum_{i=1}^{s}\omega_{1i}k_i(\sigma_2) - f_{opt}^1$, $L_2 = \sum_{i=1}^{s}\omega_{2i}k_i(\sigma_1) - f_{opt}^2$, for $\forall\sigma\in\Omega$ $\Delta_1 = \sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1$, $\Delta_2 = \sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2$. Then, if $\exists\sigma\in\Omega$ for which $a_1\Delta_1 + a_2\Delta_2 < \min(a_1L_1, a_2L_2)$, then $\sigma_1 \vee \sigma_2 \notin \sigma^2(a_1, a_2)$. If there is no such feasible solution, then $\sigma^2(a_1, a_2) = \sigma_1$ if $\min(a_1L_1, a_2L_2) = a_2L_2$ or $\sigma^2(a_1, a_2) = \sigma_2$ if

$min(a_1L_1, a_2L_2) = a_1L_1$. If $a_1 = 1,\ a_2 = 1$, then $\sigma_1 \vee \sigma_2 \notin \sigma^2(a_1,\ a_2)$ if $\exists\sigma \in \Omega$ for which $\Delta_1 + \Delta_2 < min(L_1, L_2)$.

For convenience, we introduce the following notation:

$\sigma(\Delta_1, \Delta_2)$ is a feasible solution $\sigma \in \Omega$ for which $\Delta_1 = \sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1$ and $\Delta_2 = \sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2$. $\sigma(a_1, a_2, \Delta_1, \Delta_2)$ is some $\sigma \in \Omega$ such that $\sigma \in \sigma^2(a_1,\ a_2)$ for given coefficients $a_1 > 0,\ a_2 > 0$ and for which $\Delta_1 = \sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1$, $\Delta_2 = \sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2$.

Corollary 3. We can solve the problem (10) for cases $\forall a_1 > 0$, $a_2 = 1$ or $a_1 = 1$, $\forall a_2 > 0$. Indeed,

$$\arg\min_{\sigma\in\Omega}\left\{a_1\left[\sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1\right] + a_2\left[\sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2\right]\right\} =$$

$$= \arg\min_{\sigma\in\Omega}\left\{\left[\sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1\right] + \frac{a_2}{a_1}\left[\sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2\right]\right\} =$$

$$= \arg\min_{\sigma\in\Omega}\left\{\frac{a_1}{a_2}\left[\sum_{i=1}^{s}\omega_{1i}k_i(\sigma) - f_{opt}^1\right] + \left[\sum_{i=1}^{s}\omega_{2i}k_i(\sigma) - f_{opt}^2\right]\right\}.$$

Corollary 4. If the efficiency of a solving method for the problem (1) does not depend on the signs of the coefficients $\omega_i,\ i = \overline{1, s}$, then we can reformulate Statement 1 in an obvious way for the combinatorial optimization problem of the form

$$\max_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_i k_i(\sigma). \tag{11}$$

Indeed,

$$\arg\max_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_i k_i(\sigma) = \arg\min_{\sigma\in\Omega}\sum_{i=1}^{s}(-\omega_i)k_i(\sigma).$$

Corollary 5. Suppose that an efficient algorithm for the problem (1) exists only for $\forall\omega_i > 0,\ i = \overline{1, s}$. Then we reformulate Statement 1 in an obvious way for the problem

$$\max_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_i k_i(\sigma),\ \omega_i > 0,\ i = \overline{1, s}.$$

$$\arg\max_{\sigma\in\Omega}\left\{a_1\left[f_{opt}^1 - \sum_{i=1}^{s}\omega_{1i}k_i(\sigma)\right] + a_2\left[f_{opt}^2 - \sum_{i=1}^{s}\omega_{2i}k_i(\sigma)\right]\right\} =$$

$$= \arg\max_{\sigma\in\Omega}\sum_{i=1}^{s}(a_1\omega_{1i} + a_2\omega_{2i})k_i(\sigma), \quad a_1 > 0,\ a_2 > 0.$$

$$f_{opt}^1 = \max_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_{1i}k_i(\sigma),\ \ f_{opt}^2 = \max_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_{2i}k_i(\sigma).$$

Statement 2. Suppose that $\sigma(a_1, a_2, \Delta_1, \Delta_2) \neq \sigma_1 \vee \sigma_2$, $\exists\sigma(a_1 + \Delta a_1, a_2, \Delta_1', \Delta_2') \neq \sigma_1 \vee \sigma_2$, and $\Delta_1' \neq \Delta_1$. Then, $\Delta a_1(\Delta_1' - \Delta_1) < 0$, $(\Delta_1' - \Delta_1)(\Delta_2' - \Delta_2) < 0$.

Proof of Statement 2 by contradiction. Suppose $\Delta a_1 > 0$ and $\Delta_1' > \Delta_1$. Since $\sigma(a_1 + \Delta a_1, a_2, \Delta_1', \Delta_2') \neq \sigma_1 \vee \sigma_2$, then $(a_1 + \Delta a_1)\Delta_1' + a_2\Delta_2' < \min(a_1L_1, a_2L_2)$. Then, $\Delta_2' < \Delta_2$ because otherwise we have $(a_1 + \Delta a_1)\Delta_1 + a_2\Delta_2 < < (a_1 + \Delta a_1)\Delta_1' + a_2\Delta_2'$ which is impossible since $\sigma(a_1 + \Delta a_1, a_2, \Delta_1', \Delta_2') \in \in \sigma^2(a_1 + \Delta a_1, a_2)$ and therefore $(a_1 + \Delta a_1)\Delta_1 + a_2\Delta_2 \geq (a_1 + \Delta a_1)\Delta_1' + a_2\Delta_2'$. Since $(a_1 + \Delta a_1)\Delta_1 + a_2\Delta_2 \geq (a_1 + \Delta a_1)\Delta_1' + a_2\Delta_2'$, we have $(a_1 + \Delta a_1)(\Delta_1' - \Delta_1) \leq a_2(\Delta_2 - \Delta_2')$ and $a_1(\Delta_1' - \Delta_1) < a_2(\Delta_2 - \Delta_2')$. Then, $a_1\Delta_1' + a_2\Delta_2' < a_1\Delta_1 + a_2\Delta_2$, which is impossible because $\sigma(a_1, a_2, \Delta_1, \Delta_2) \in \sigma^2(a_1, a_2)$. Consequently, $\Delta_1' < \Delta_1$ and $\Delta_2' > \Delta_2$ (if $\Delta_2' \leq \Delta_2$, then $a_1\Delta_1' + a_2\Delta_2' < a_1\Delta_1 + a_2\Delta_2$ which is impossible). We can similarly prove that $\Delta_1' > \Delta_1$ and $\Delta_2' < \Delta_2$ if $\Delta a_1 < 0$. This completes the proof.

Corollary. Symmetric formulas are true for $\sigma(a_1, a_2 + \Delta a_2, \Delta_1', \Delta_2') \neq \sigma_1 \vee \sigma_2$, $\Delta_2' \neq \Delta_2$, namely: $\Delta a_2(\Delta_2' - \Delta_2) < 0$, $(\Delta_2' - \Delta_2)(\Delta_1' - \Delta_1) < 0$.

Statement 3. Suppose that $a_1L_1 > a_2L_2$ and $\sigma(a_1, a_2, \Delta_1, \Delta_2) = \sigma_1$. Then $\sigma^2(a_1', a_2) = \sigma_1$ for $\forall a_1' > a_1$.

Proof by contradiction. Suppose that $\sigma(a_1', a_2, \Delta_1', \Delta_2') \neq \sigma_1$ ($min(a_1L_1, a_2L_2) = a_2L_2$). Therefore, $a_1'\Delta_1' + a_2\Delta_2' \leq a_2L_2$. Hence, $a_1\Delta_1' + + a_2\Delta_2' < a_1'\Delta_1' + a_2\Delta_2'$, since $a_1 < a_1'$. Therefore, $\sigma(\Delta_1, \Delta_2) \notin \sigma^2(a_1, a_2)$.

Corollary 1. If $\sigma^2(a_1, a_2) = \sigma_1$, then an increase in the first coefficient should not exceed the value of a_1 at a fixed a_2.

Corollary 2. Similar result holds for the case $a_2L_2 > a_1L_1$, $a_1 = const$, coefficient a_2 increases.

Statement 4. If $\sigma(a_1, a_2, \Delta_1, \Delta_2)$ satisfies the conditions $\Delta_1 > l_1$, $\Delta_2 > l_2$ for any $a_1 > 0$, $a_2 > 0$, then there is no feasible $\sigma(\Delta_1', \Delta_2')$ for which $\Delta_1' \leq l_1$, $\Delta_2' \leq l_2$.

Proof by contradiction. In this case, $a_1\Delta_1' + a_2\Delta_2' < a_1\Delta_1 + a_2\Delta_2$ and $\sigma(\Delta_1, \Delta_2) \notin \sigma^2(a_1, a_2)$.

Algorithms to solve the combinatorial optimization problem in the first formulation under uncertainty

1) The first condition: to find a feasible solution □ corresponding to $min(\Delta_1 + \Delta_2)$. We determine $\sigma^2(1, 1)$. If $\sigma^2(1, 1) \neq \sigma_1 \vee \sigma_2$, then the obtained solution $\sigma(1, 1, \Delta_1, \Delta_2) \in \sigma^2(1, 1)$ has $\Delta_1 + \Delta_2 < min(L_1, L_2)$ which is the minimum possible. Otherwise, the optimal solution according to this criterion is $\sigma_1 \vee \sigma_2$.

2) The second condition: to find a feasible solution □ for which

$$\Delta_1 \leq l_1,\ \Delta_2 \leq l_2. \qquad (12)$$

a) Suppose that $\sigma^2(1, 1) \neq \sigma_1 \vee \sigma_2$ (i.e., we have found $\sigma(1, 1, \Delta_1, \Delta_2)$ with $\Delta_1 + \Delta_2 < min(L_1, L_2)$) and $\sigma(1, 1, \Delta_1, \Delta_2)$ does not satisfy (12). Then, by virtue of Statements 2 and 3, we sequentially build solutions $\sigma(a_1, a_2, \Delta_1, \Delta_2)$, at each step first increasing a_1 at $a_2 = const$, then increasing a_2 at $a_1 = const$. As a result, we will either obtain a solution □ satisfying (12) or will build a set of solutions each of which does not satisfy (12). Then we select a compromise solution for which we have

$$min \sum_t C_{jt}(\Delta_{jt} - l_{jt}), \quad \forall t \ \Delta_{jt} > l_{jt}$$

where $C_j > 0, \ j = \overline{1, s}$ are expert coefficients.

b) Suppose that $\sigma(1, 1, \Delta_1, \Delta_2) = \sigma_1 \vee \sigma_2$. Since $l_1 < L_1, l_2 < L_2$, then $\sigma(1, 1, \Delta_1, \Delta_2) = \sigma_1 \vee \sigma_2$ does not satisfy the condition (12). Then, if $L_1 < L_2$, we find $\sigma^2(a_1, 1)$ where $a_1 = L_2/L_1$. $\sigma^2(a_1, 1) \neq \sigma_1 \vee \sigma_2$ if there exists $\sigma(\Delta_1, \Delta_2)$ for which $a_1\Delta_1 + \Delta_2 < L_2$. Suppose that $L_1 > L_2$, then we find $\sigma^2(1, a_2)$, $a_2 = L_1/L_2$, and obtain a non-trivial solution if $\exists\sigma(\Delta_1, \Delta_2)$ such that $\Delta_1 + a_2\Delta_2 < L_1$. Suppose that we have found a non-trivial solution $\sigma(\Delta_1, \Delta_2) \neq \sigma_1 \vee \sigma_2$. If $\Delta_1 \leq l_1, \Delta_2 \leq l_2$, then we have a solution. Suppose that $\Delta_1 \leq l_1 \vee \Delta_2 \leq l_2$ (if $\Delta_1 > l_1, \Delta_2 > l_2$, then there is no □ that satisfies (12)). Then, in accordance with Statements 2 and 3, we can organize an iterative procedure for obtaining solutions $\sigma(a_1, a_2, \Delta_1, \Delta_2)$ by sequential increase of a_1 at $a_2 = const$ and, vice versa, increasing sequentially a_2 at $a_1 = const$. As a result, we will either obtain a solution $\sigma(\Delta_1, \Delta_2)$ satisfying the condition (12) or a solution that violates the condition (12) as it minimally possible (see item (a)).

c) If $\sigma(1, 1) = \sigma_1 \vee \sigma_2$, $\sigma(a_1, a_2)$ from item (b) is equal to $\sigma_1 \vee \sigma_2$, then we choose a compromise from $\{\sigma_1, \sigma_2\}$ as a solution.

3) We can satisfy the condition $min\,\Delta_1, \ \Delta_2 \leq l_2$ or $min\,\Delta_2, \ \Delta_1 \leq l_1$ as a result of the implementation of the iterative procedure given in item 2 (a) or item 2 (b), since for any $\sigma(\Delta_1, \Delta_2) \in \sigma^2(a_1, a_2)$ we have: $\nexists\sigma(\Delta_1', \Delta_2')$ for which $\Delta_1' \leq \Delta_1, \ \Delta_2' \leq \Delta_2$.

We can find a feasible solution that satisfies conditions 4) or 5), that is $\min_{\sigma\in\Omega}(a_1\Delta_1 + a_2\Delta_2)$ or $\min_{\sigma\in\Omega}\{M\Delta = P_1\Delta_1 + P_2\Delta_2\}$, in an obvious way by solving one combinatorial optimization problem of the form (1).

Remark 3. We modify the above algorithms in an obvious way for the case of Statement 1, Corollary 5.

Combinatorial optimization under uncertainty. The second formulation

There are L sets of weights $\{\omega_i^l, \ i = \overline{1, s}\}, \ l = \overline{1, L}$. Each one may be a set of coefficients $\omega_1, \ldots, \omega_s$ of the problem (1) at the stage of fulfillment of its solution. We can specify probabilities $P_l > 0, \ l = \overline{1, L}$, $\sum_l P_l = 1$, for each of the possible sets of weights (such probabilities do not exist if the uncertainty is not described by a probabilistic model). We need to find a feasible solution $\sigma \in \Omega$ that satisfies one of the following conditions:

1) Let us denote: $f^l_{opt} = \min_{\sigma\in\Omega} \sum_{i=1}^{s} \omega^l_i k_i(\sigma)$, $\{\sigma_l\} = \arg\min_{\sigma\in\Omega} \sum_{i=1}^{s} \omega^l_i k_i(\sigma)$, $L_l = \sum_{\substack{m=1 \\ m\neq l}}^{L} \left(\sum_{i=1}^{s} \omega^m_i k_i(\sigma_l) - f^m_{opt} \right)$.

Remark 4. If $\{\sigma_l\}$ consists of more than one solution, we keep the one on which we have $\min_{\{\sigma_l\}} L_l$ and denote this solution by σ_l (we show below how to obtain σ_l for the case when σ is finite).

Suppose that $L_p = \min_l L_l$ (L_p corresponds to a solution σ_p). We need to find σ that reaches $\min_{\sigma\in\Omega} \sum_{l=1}^{L} \left(\sum_{i=1}^{s} \omega^l_i k_i(\sigma) - f^l_{opt} \right)$.

2) Find a feasible solution $\sigma(\Delta_1, \ldots, \Delta_L)$ for which

$$\Delta_i \le l_i,\ l_i > 0,\ i = \overline{1, L}. \tag{13}$$

3) Let us introduce a random variable $F = \sum_{i=1}^{s} \overline{\omega}_i k_i(\sigma) - \overline{f}_{opt}$ where s+1-dimensional discrete random variable $\overline{\omega}_1 \ldots, \overline{\omega}_s, \overline{f}_{opt}$ is specified by the table:

$$\left\{ \begin{matrix} \omega^l_1, \ldots, \omega^l_s,\ f^l_{opt} \\ P_l > 0,\ l = \overline{1, L} \end{matrix} \right\}.$$

We need to find a solution to this problem:

$$\min_{\sigma\in\Omega} MF = \min_{\sigma\in\Omega} \sum_{l=1}^{L} P_l \left(\sum_{i=1}^{s} \omega^l_i k_i(\sigma) - f^l_{opt} \right).$$

4) Find a feasible solution that satisfies the condition

$$\min_{\sigma\in\Omega} \sum_{l=1}^{L} a_l \left(\sum_{i=1}^{s} \omega^l_i k_i(\sigma) - f^l_{opt} \right)$$

where $\forall a_l > 0$ are the coefficients set by experts.

Finding a solution that satisfies one of the four given conditions follows from the following Statement 5 which is a natural generalization of Statement 1.

Statement 5. The following is true for arbitrary $a_l > 0,\ l = \overline{1, L}$:

$$\arg\min_{\sigma\in\Omega} \sum_{l=1}^{L} a_l \left[\sum_{i=1}^{s} \omega^l_i k_i(\sigma) - f^l_{opt} \right] = \arg\min_{\sigma\in\Omega} \sum_{i=1}^{s} \left(\sum_{l=1}^{L} a_l \omega^l_i \right) k_i(\sigma). \tag{15}$$

Proof of Statement 5 is similar to the proof of Statement 1.

Corollary 1. Solving of the problem $\min_{\sigma\in\Omega} \sum_{l=1}^{L} a_l \left[\sum_{i=1}^{s} \omega^l_i k_i(\sigma) - f^l_{opt} \right]$ reduces to solving of one problem of the form (1):

$$\min_{\sigma\in\Omega}\sum_{i=1}^{s}\left(\sum_{l=1}^{L}a_l\omega_i^l\right)k_i(\sigma). \qquad (16)$$

Corollary 2. Suppose that a solution $\sigma(a_1,\ldots,a_L,\Delta_1,\ldots,\Delta_L)\neq\ \neq\sigma_1\vee\ldots\vee\sigma_L$ belongs to the set $\sigma^2(a_1,\ldots,a_L)$. Then, $\nexists\sigma(\Delta_1',\ldots,\Delta_L')\in\Omega$ for which $\Delta_i'\le\Delta_i$, $i=\overline{1,L}$, $\Delta^T=(\Delta_1,\ldots,\Delta_L)\neq\Delta'^T=(\Delta_1',\ldots,\Delta_L')$.

Corollary 3. To obtain a solution according to condition 1), it is necessary to set $a_l=1,\ l=\overline{1,L}$ in the problem (1). In this case, the solution to problem (15) does not coincide with the solution σ_p corresponding to $\min_l L_l$ if there is such a feasible solution $\sigma(\Delta_1,\ldots,\Delta_L)$, $\Delta_l=\sum_{i=1}^{s}\omega_i^l k_i(\sigma)-f_{opt}^l$ for which

$$\sum_{l=1}^{L}\Delta_l<L_p. \qquad (17)$$

If there is no such a feasible solution $\sigma(\Delta_1,\ldots,\Delta_L)$ that $\sum_{l=1}^{L}\Delta_l<L_p$, then $\sigma(1,\ldots,1)=\sigma_p$. So, without knowing the sets $\{\sigma_1\},\ldots,\{\sigma_L\}$ we automatically obtain σ_p. At L = 2, solving two problems $a_1>0$ (a sufficiently large number), $a_2=1$ and $a_1=1$, $a_2>0$ (a sufficiently large number), we find $\sigma_1,\ \sigma_2$ that correspond to the minimum possible $L_1,\ L_2$ in case when σ is finite. In the general case, if Ω is finite, $\sigma_j=\sigma^2(1,\ldots,1,a_j,1,\ldots,1)$ for sufficiently large a_j.

Corollary 4. Suppose that $\sigma(a_1,\ldots,a_L,\Delta_1,\ldots,\Delta_L)\neq\sigma_1\vee\ldots\vee\sigma_L$ and $\exists\sigma(a_1,a_2,\ldots,a_{i-1},a_i',a_{i+1},\ldots,a_L,\Delta_1',\ldots,\Delta_L')\neq\sigma_1\vee\ldots\vee\sigma_L$, $a_i'\neq a_i$. Then an analogue of the Statement 2 is true:

$$(a_i'-a_i)(\Delta_i'-\Delta_i)<0,\ (\Delta_i'-\Delta_i)[(a_1\Delta_1'+\ldots+a_{i-1}\Delta_{i-1}'+a_{i+1}\Delta_{i+1}'+a_L\Delta_L')-$$
$$-(a_1\Delta_1+\ldots+a_{i-1}\Delta_{i-1}+a_{i+1}\Delta_{i+1}+a_L\Delta_L)]<0. \qquad (19)$$

Proof of this inequality almost literally repeats the proof of Statement 2.

Corollary 5. Suppose that $\sigma(1,\ldots,1,\Delta_1,\ldots,\Delta_L)\neq\sigma_p$ and does not satisfy the condition 2). Then, by logic of the inequalities (19), we can organize a sequential procedure for the problem (16) solving by increasing $\forall a_i$ at each iteration if $\Delta_i>l_i$ and decreasing $\forall a_j$ if $\Delta_j<l_j$. As a result, we either find a solution □ satisfying the condition 2) or obtain a set of solutions $\{\sigma\}^1$ each of which violates the condition 2). It is true that $\sigma(1,\ldots,1,\Delta_1,\ldots,\Delta_L)\in\{\sigma\}^1$. Denote $\{\sigma\}^2=\{\sigma_1,\ldots,\sigma_L\}$. Then a compromise solution for the condition 2) is a solution $\overline{\sigma}\in\{\sigma\}^1\cup\{\sigma\}^2$ that reaches

$$\min\sum_t C_{jt}(\Delta_{jt}-l_{jt}),\quad \forall t\ \Delta_{jt}>l_{jt}$$

where $C_j>0,\ j=\overline{1,s}$ are expert coefficients.

Corollary 6. We can find a feasible solution σ satisfying the condition 3) or 4) by solving one problem of the form (16).

Corollary 7 (analogue of Corollary 5 to Statement 1). We can solve similarly in the obvious way the combinatorial optimization problem under uncertainty of the form

$$\max_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_i k_i(\sigma),\ \omega_i > 0,\ i=\overline{1,s}. \quad (18)$$

Corollary 8. If the efficiency of the solving method for the problem (1) does not depend on the signs of the coefficients ω_i, then we can similarly solve the following vector optimization problem: find a compromise solution for the problem

$$\min_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_i^l k_i(\sigma),\ l=\overline{1,L},\ \max_{\sigma\in\Omega}\sum_{i=1}^{s}\omega_i^p k_i(\sigma),\ p=\overline{1,L}.$$

CONCLUSIONS

We formulate a sufficiently common class of combinatorial optimization problems under uncertainty. The latter means a possible ambiguity of the values of coefficients included in the optimality criterion at the time of fulfillment of the optimal solution. We also formulate criteria for a compromise solution obtaining and algorithms for its finding based on the well-known idea of linear convolution of criteria [3–7]. A distinctive feature of the presented algorithms is that their efficiency is unambiguously determined by the efficiency of solving the combinatorial optimization problem in a deterministic formulation.

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